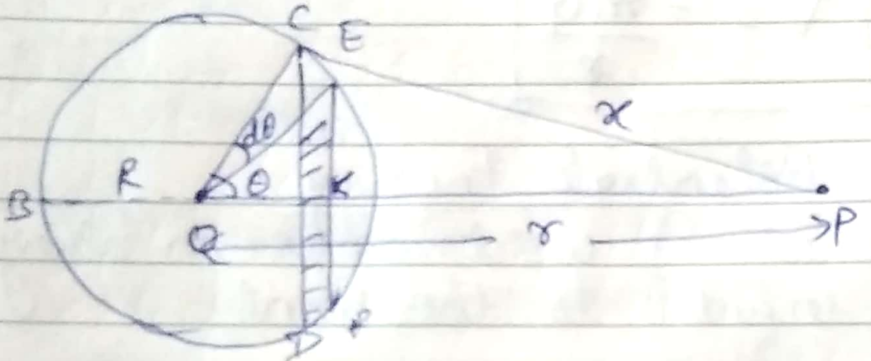
Gravitational potential & field due to spherical shell:

(a) Gravitational Potential

(i) Point outside the shell



point P is at distance r from the center of a spherical shell of mass M and surface density σ

Let us consider a slice CEFD in the form of a ring of planes CD & CE.

Note: A spherical shell consists of 'n' no. of rings piled up together

radius of ring EK = OE sin θ = R sin θ. — (1)

Circumference of the ring = 2π R sin θ — (2)

width of CE = R dθ — (3) [∵ Angle = $\frac{\text{Arc}}{\text{radius}}$

Surface area of the ring = circumference × width

$= 2\pi R \sin\theta \cdot R\,d\theta = 2\pi R^2 \sin\theta\, d\theta$ — (4)

Mass = density × surface area

$(dM) = 2\pi R^2 \sin\theta \cdot d\theta \cdot \sigma$ — (5)

Every point on the ring is suppose at a distance 'x' from point P.

small Potential at Point P due to this ring From equation (1), (2), (3), (4) & (5)

$dV = -\frac{\text{mass of slice } G}{x} = -\frac{dM}{x}$ $\left[\because V = \frac{-MG}{R}\right.$ as proved earlier$\left.\right]$

$dV = -\frac{2\pi R^2 \sin\theta\, d\theta\, \sigma\, G}{x}$ — (6)

In ΔOEP

$EP^2 = (OE)^2 + (OP)^2 - 2OE \cdot OP \cos\theta$

$x^2 = R^2 + r^2 - 2Rr\cos\theta$ — (7)

Differentiating equation (7)

$2x\,dx = 0 + 0 + 2Rr\sin\theta\, d\theta$ — (8) $\left[\because R \,\&\, r \text{ are constant}\right]$

Putting equation (8) in equation (6)

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$$dN = \frac{2\pi R^2 \sin\theta\, d\theta\, \sigma}{R r \sin\theta\, d\theta} G\, dx$$

$$dN = -\frac{2\pi R \sigma G}{r} dx \quad \text{— (9)}$$

Rings (slices) will be covered between A & B on either sides of center.
Extremums on either side at point A is $(r-R)$ & point B is $(r+R)$ so Integrating equation '9' between these extremums.

$$V = -\int_{r-R}^{r+R} \frac{2\pi R \sigma G}{r} dx \quad \text{— (10)}$$

$$= -\frac{2\pi R \sigma G}{r} \int_{r-R}^{r+R} dx = -\frac{2\pi R \sigma G}{r} \left[x \right]_{r-R}^{r+R}$$

$$= -\frac{2\pi R \sigma G}{r} \left[\cancel{r} + R - \cancel{r} + R \right]$$

$$= -\frac{2\pi R \sigma G}{r} 2R = -\frac{4\pi R^2 \sigma G}{r}$$

$\because$ Total Mass $M = 4\pi R^2 \sigma$

$\therefore$ $\boxed{V = -\frac{MG}{r}}$ ✓ — (11)

(ii) <u>on the surface of the shell</u>

putting $r = R$ in equation (11).

we get,

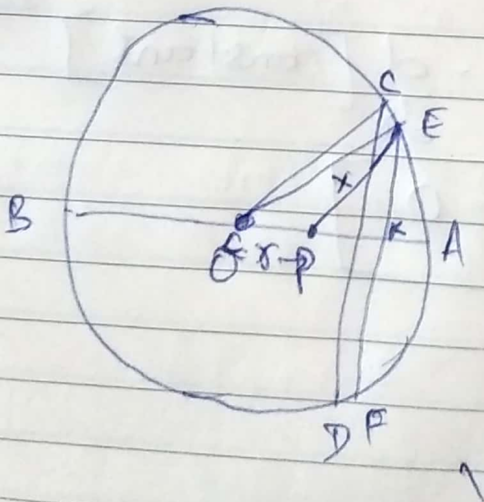
$$\boxed{V = -\frac{MG}{R}}$$

or Integrating eqn (10) with limits 0 to 2R

$$V = -\int_0^{2R} \frac{2\pi R\sigma G}{r}\,dx$$

$$\boxed{V = -\frac{MG}{R}}$$

(iii) <u>At a point inside the shell.</u>



The only change in this part from part(i) is that the limits will change to (R−r to R+r)

from Eqn (10)

$$V = -\int_{R-r}^{R+r} \frac{2\pi R\sigma G}{r}$$

$$V = -\frac{2\pi R\sigma G}{r}\int dx = -\frac{MG}{R} \quad \therefore \boxed{V = -\frac{MG}{R}}$$

(b) Gravitational Field

(i) point outside the shell

$$E = -\frac{dV}{dr} = -\frac{d}{dr}\left(-\frac{MG}{r}\right) = -\frac{MG}{r^2}$$

(ii) ~~on the~~ On the surface

$$\boxed{E = -\frac{dV}{dr}\Big|_{r=R} = -\frac{MG}{R^2}}$$

(iii) A point inside the shell

∵ potential inside the shell

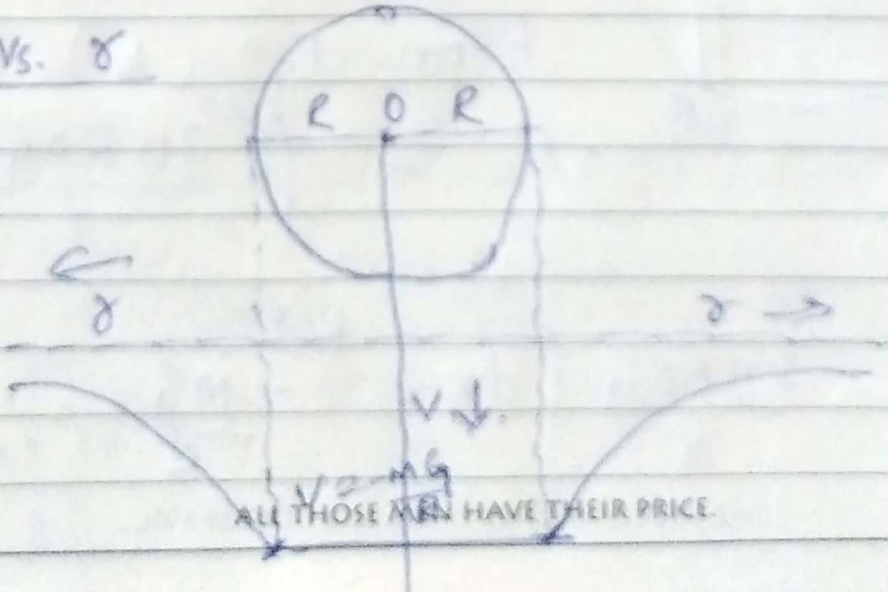
$$V = -\frac{MG}{R} = \text{constant}$$

$$\therefore E = -\frac{dV}{dr} = -\frac{d}{dr}[\text{constant}]$$

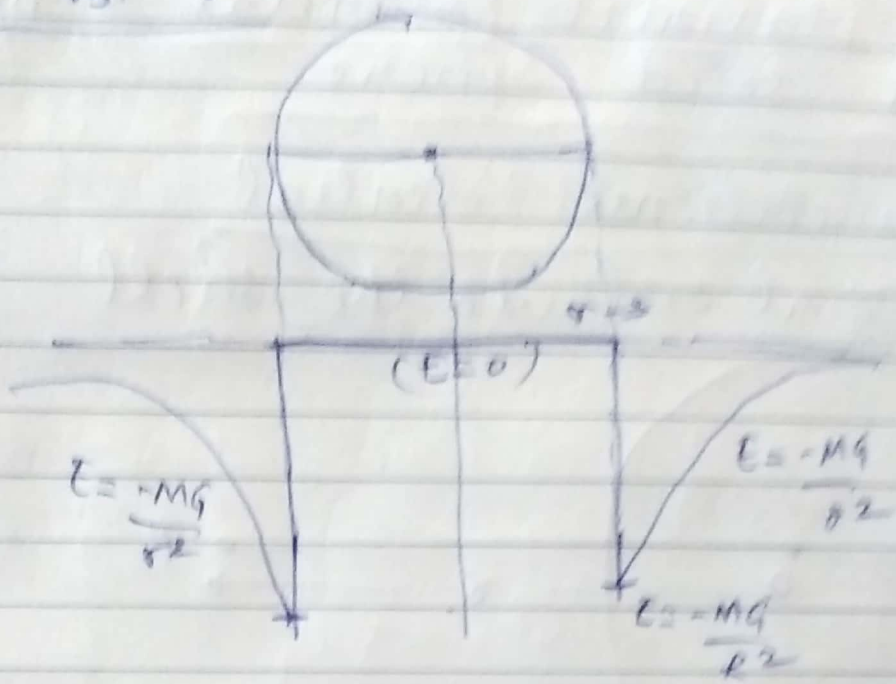
$$\boxed{E = 0}$$

Graph

V vs. r



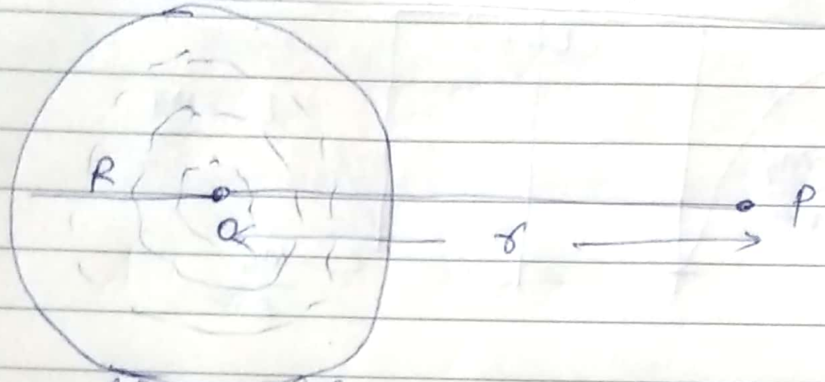
E vs. r



Gravitational Potential & field due to a solid sphere.

(1) Gravitational Potential

(i) point outside the shell



A solid sphere can be imagined as no. of spherical shells piled up one above the other.

If it consists of 'n' no. of shells with each having masses $m_1, m_2, m_3 - - - m_n$ respectively.

As proved in previous section, the gravitational potential at a point P due to a spherical shell is

$$V_1 = -\frac{m_1 G}{r} \quad \text{--- (1)}$$

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$\therefore$ Total Gravitational potential

$$V = V_1 + V_2 + - - - V_n.$$

$$= -\frac{m_1 G}{r} - \frac{m_2 G}{r} - - - - \frac{m_n G}{r}$$

$$= -\frac{G}{r}(m_1 + m_2 + - - - + m_n) \quad \text{--- (2)}$$

$m_1 + m_2 + \cdots + m_n = M$ (Total mass of solid sphere) —— (3)

putting equation (3) in equation (2)

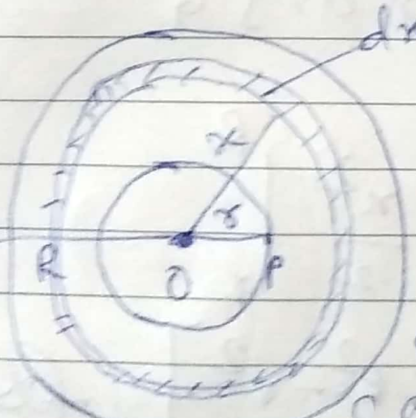
$$\boxed{V = -\frac{MG}{r}}$$

(ii) On the surface of solid sphere

$r = R$

$$\boxed{V = -\frac{MG}{R}}$$

(iii) Inside the solid sphere



Solid sphere can be considered as having two parts. One the inner solid sphere of radius 'r' and other part has concentric cells with it of radius ranging from r to R

potential at point P due to inner solid sphere of radius r

$$= -\frac{\text{mass of sphere } G}{r} = \frac{-\frac{4}{3}\pi r^3 \rho G}{r} = -\frac{4\pi r^2 \rho G}{3} \quad \text{—— (1)}$$

where ρ = density of the solid.

Potential at P due to the concentric spherical shells, let us consider a shell of thickness dx and is at distance 'x' from the centre 'o'

$$dV_0 = -\frac{\text{mass of shell } G}{x} \quad \text{— (2)}$$

$$\text{Mass of shell 'dm'} = \underbrace{4\pi x^2 \, dx}_{\text{volume}} \, \rho \quad \text{— (3)}$$

$$dV_0 = -\frac{4\pi x^2 \, dx \, \rho \, G}{x} \quad \text{— (4)}$$

$$= -4\pi x \, dx \, \rho \, G$$

Integrating from r to R to cover all the spherical shells

$$V_0 = -\int_r^R 4\pi x \, dx \, \rho \, G$$

$$= -4\pi \rho G \left. \frac{x^2}{2} \right|_r^R$$

$$= -\frac{4\pi \rho G}{2} \left[R^2 - r^2 \right] \quad \text{— (5)}$$

$$= -\frac{4}{3}\pi \rho G \, \frac{3(R^2 - r^2)}{2} \quad \text{— (5)}$$

Total potential (Adding eq^n (1) & (5)]

$$= -\frac{4}{3}\pi r^2 \rho G - \frac{4}{3}\pi \rho G \left(\frac{3R^2 - 3r^2}{2} \right)$$

$$= -\frac{4}{3}\pi \rho G \left[r^2 + \frac{3R^2}{2} - \frac{3r^2}{2} \right]$$

$$= -\frac{4}{3}\pi \rho G \left[\frac{3R^2}{2} - \frac{r^2}{2} \right]$$

Dividing & Multiplying by R^3

$$= -\frac{4}{3}\pi R^3 \rho G \left[\frac{3R^2}{2R^3} - \frac{r^2}{2R^3} \right]$$

$$\boxed{V = -\frac{MG}{2R^3}(3R^2 - r^2)}$$

At the center of solid sphere

$r = 0$

$$V = -\frac{MG}{2R^3} 3R^2 = -\frac{3}{2}\frac{MG}{R}$$

$$\boxed{V = -\frac{3}{2}\frac{MG}{R}}$$

Gravitational Field

(i) outside the shell

$$V = -\frac{MG}{r}$$

$$E = -\frac{dV}{dr} = -\frac{d}{dr}\left(\frac{-MG}{r}\right)$$

$$\boxed{E = -\frac{MG}{r^2}}$$

(ii) on the surface

$r = R$

$$\boxed{E = \frac{-MG}{R^2}}$$

inside

$r = 0$

$E = 0$.

(iii) Inside the sphere

$$V = -\frac{MG}{2R^3}\left(3R^2 - r^2\right)$$

$$E = -\frac{dV}{dr} = -\frac{d}{dr}\left[-\frac{MG}{2R^3}\left(3R^2 - r^2\right)\right]$$

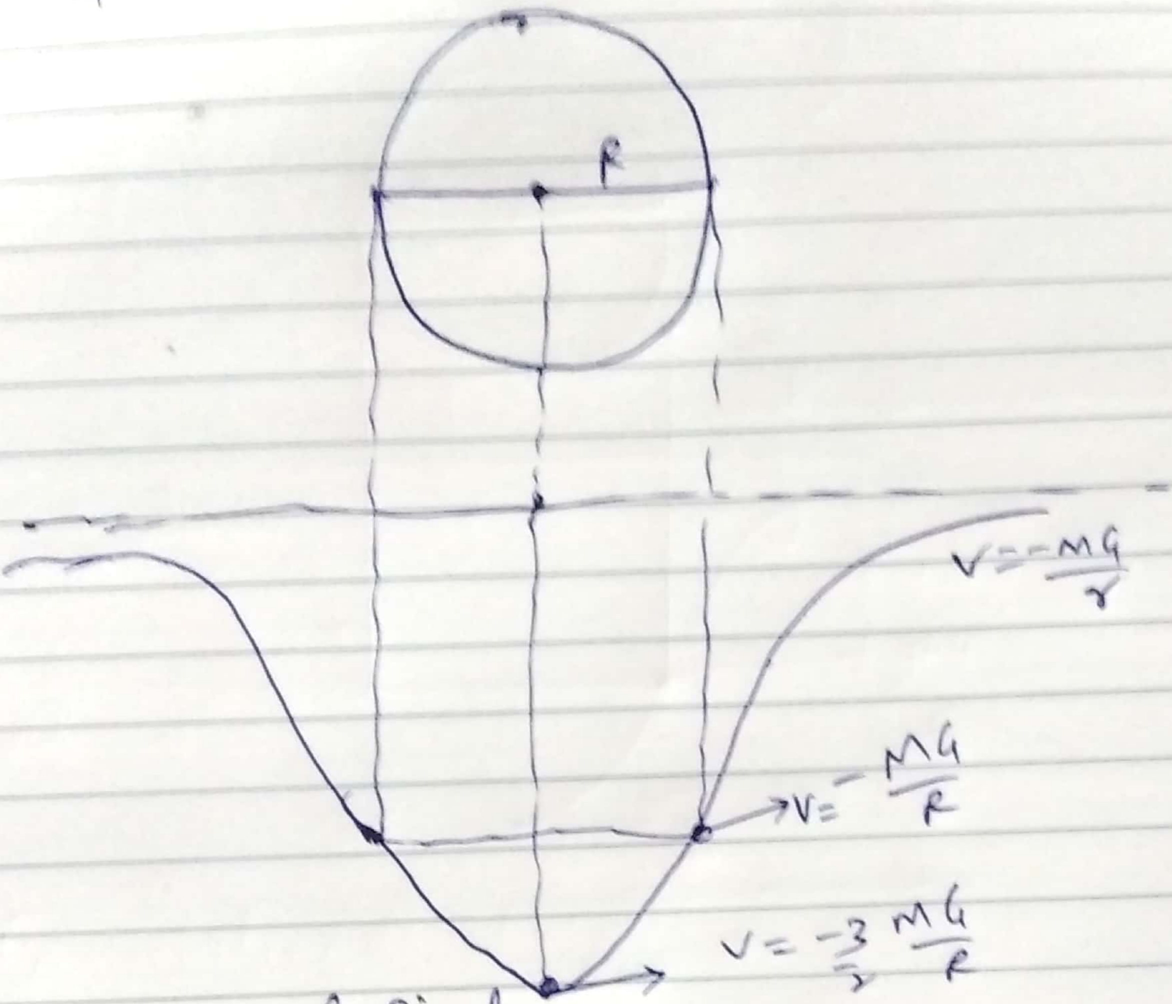
$$= -\frac{MG}{\cancel{2}R^3} \cdot \cancel{2}r$$

$$\boxed{E = -\frac{MG}{R^3}r}$$

$E \propto r$

Graph

Gravitational potential due to a solid sphere



Gravitational field